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SOLUTION OF A PLANE STEFAN PROBLEM FOR A HALF-SPACE BY THE METHOD OF DEGENERATE HYPERGEOMETRIC TRANSFORMATIONS

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A method is given for constructing the analytic solution of a plane nonstationary Stefan problem.

Analytical methods of solving multidimensional nonstationary Stefan problems have only started to be produced. Methods existing earlier for the solution of such problems ([1, 2], etc.) were quite approximate in nature. The general solution of a quasistationary plane Stefan problem is obtained in [3]. An analytical method of solving a nonstationary plane Stefan problem is proposed in this paper for a half-space in application to the process of freezing the ground bounded on one side by a plane and extending without limit to the other side.

Let us consider the problem on the dynamics of the freezing and cooling zones (zones I and II) of ground under a plane source of cold located on the surface of a semiinfinite medium (ground) (Fig. 1). The general formulation of such a problem with two moving boundaries is described by the following system of equations and boundary conditions:

$$\frac{\partial t_k(x_1, x_2, \tau)}{\partial \tau} = a_k \sum_{i=1}^{2} \frac{\partial^2 t_k(x_1, x_2, \tau)}{\partial x_i^2}, \quad k = 1, 2, \tag{1}$$

for $k = 1$, $(x_1, x_2) \in D_{x,1} = \{|x_1| < \xi_1(\tau),\ 0 < x_2 < \xi(x_1, \tau)\}$;

for $k = 2$, $(x_1, x_2) \in D_{x,2} = D_x^{(1)} + D_x^{(2)}$; $D_x^{(1)} = \{|x_1| \leqslant \xi_1(\tau),$ $\xi(x_1, \tau) < x_2 < \nu(x_1, \tau)\}$; $D_x^{(2)} = \{|x_1| \geqslant \xi_1(\tau),\ |x_1| < \nu_1(\tau);$

$$0 < x_2 < \nu(x_1, \tau)\}; \quad \tau > 0;$$

$$t_k(x_1, x_2, 0) = f_k(x_1, x_2); \tag{2}$$

$$t_1(x_1, 0, \tau) = \varphi_1(x_1, \tau) \quad \text{for} \quad |x_1| \leqslant \xi_1(\tau); \tag{3}$$

on $S_{x,0} = \{x_2 = 0,\ |x_1| \geqslant \xi_1(\tau),\ |x_1| \leqslant \nu_1(\tau)\}$

$$t_2(x_1, x_2, \tau) = \varphi_2(x_1, \tau); \tag{4}$$

on $S_{\tau,1} = \{x_2 = \xi(x_1, \tau),\ |x_1| \leqslant \xi_1(\tau)\}$

$$t_k(x_1, x_2, \tau) = 0 \tag{5}$$

and

$$\sum_{i=1}^{2} \left(\lambda_1 \frac{\partial t_1(x_1, x_2, \tau)}{\partial x_i} - \lambda_2 \frac{\partial t_2(x_1, x_2, \tau)}{\partial x_i} \right) l_i = A \frac{\partial \xi(x_1, \tau)}{\partial \tau}; \tag{6}$$

on $S_{\tau,2} = \{x_2 = \nu(x_1, \tau),\ |x_1| \leqslant \nu_1(\tau)\}$

$$t_2(x_1, x_2, \tau) = f_2(x_1, x_2), \tag{7}$$

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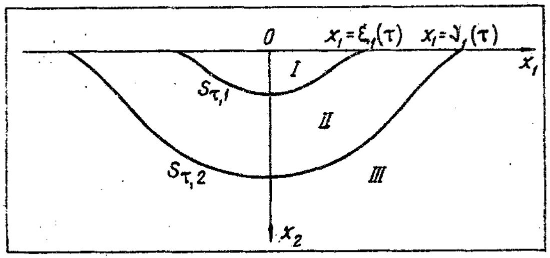

Fig. 1. Diagram of zone (I-II) location and natural temperature zone of the ground (III) for a fixed position of the boundaries $S_{\tau,1}$ and $S_{\tau,2}$.

where $f_k(x_1, x_2)$ and $\varphi_k(x_1, \tau)$ are sufficiently smooth functions of their arguments;

$$A=\sigma\gamma_w w_w; l_1=\frac{\partial\xi(x_1, \tau)}{\partial x_1}; \; l_2=1; \; \tau>0; \; k=1, 2; \; \text{for} \; |x_1|=\xi_1(\tau)$$

$$\xi(x_1, \tau)=0. \tag{8}$$

We shall seek the solution by a double application of the method of degenerate hypergeometric transformations [4, 5] under certain constraints on the shape of the moving boundaries; namely, assuming that $S_{\tau,1}$ and $S_{\tau,2}$ are smooth lines intersecting with half-lines issuing from the point (x_1 = 0, x_2 = 0) in not more than one point symmetric relative to the axis $0x_2$; $\xi(x_1, \tau) = \xi_2(\tau)\xi(x_1)$ and $\nu(x_1, \tau) = \nu_2(\tau)\nu_3(x_1)$ are single-valued functions of their arguments, $\nu_k(\tau) = a\xi_k(\tau)$; $\nu(x_1, \tau) = b\xi(x_1, \tau)$ for $x_2 > 0$; $\xi_1(0) = l_0 > 0$; $\xi(x_1, 0) \equiv \xi_0 > 0$; a and b are dimensionless proportionality factors (thermal influence factors) [6], and $2l_0$ is the minimal width of the band source of cold:

$$f_k(x_1, x_2)\equiv t_0; \; \varphi_k(x_1, \tau)=e_k\left(1-\frac{|x_1|}{\xi_1(\tau)}\right); \; e_k=\begin{cases} t_e & \text{for} \quad k=1; \\ \dfrac{t_0}{1-a} & \text{for} \quad k=2. \end{cases}$$

Using the substitution

$$x_k=y_k\xi_k(\tau), \quad k=1, 2, \tag{9}$$

and introducing the auxiliary functions

$$T_k(y_1, y_2, \tau)=t_k(x_1, x_2, \tau)+(|y_1|-1)e_k, \; k=1, 2, \tag{10}$$

we convert the problem (1)-(5) and (7) to the form

$$\frac{\partial T_k(y_1, y_2, \tau)}{\partial\tau}+e_k|y_1|\frac{\partial\ln\xi_1(\tau)}{\partial\tau}=\sum_{i=1}^{2}\xi_i^{-2}(\tau)\times$$

$$\times\left[a_k\frac{\partial^2 T_k(y_1, y_2, \tau)}{\partial y_i^2}+y_i\xi_i(\tau)\xi_i'(\tau)\frac{\partial T_k(y_1, y_2, \tau)}{\partial y_i}\right], \quad k=1, 2; \tag{11}$$

$$\text{for} \; k=1, \; (y_1, y_2)\in D_{y,1}=\{|y_1|<1, \; 0<y_2<\xi_3(x_1)\};$$

$$\text{for} \; k=2, \; (y_1, y_2)\in D_{y,2}=D_y^{(1)}+D_y^{(2)}; \; D_y^{(1)}=\{|y_1|\leqslant 1, \; 0<$$

$$<y_2<\xi_3(x_1)\}, \; D_y^{(2)}=\{|y_1|\geqslant 1, \; |y_1|<a; \; 0<y_2<a\nu_3(x_1)\};$$

$$T_k(y_1, y_2, 0)=\begin{cases} |y_1|t_e+\Delta t_e & \text{for} \quad k=1; \\ (|y_1|-a)e_2 & \text{for} \quad k=2; \; \Delta t_e=t_0-t_e; \end{cases} \tag{12}$$

$$T_k(y_1, 0, \tau)=0; \tag{13}$$

$$T_k(y_1, y_2, \tau)=0 \quad \text{for} \quad |y_1|=1; \tag{14}$$

$$T_2(y_1, y_2, \tau)=0 \quad \text{for} \quad |y_1|=a; \tag{15}$$

$$T_k(y_1, y_2, \tau)=e_k(|y_1|-1) \quad \text{for} \; |y_1|\leqslant 1, \; y_2=\xi_3(x_1); \; k=1, 2; \tag{16}$$

$$T_2(y_1, y_2, \tau) = e_2(|y_1| - a) \quad \text{for} \quad |y_1| \leqslant a,\ y_2 = a\nu_3(x_1),\ \tau > 0. \tag{17}$$

We seek the solution of the problem (11)-(17) sequentially by applying the method of degenerate hypergeometric transformations in y_1 and $y_3 = y_2/\xi_3(x_1)$:

$$U_k(\gamma_1, y_2, \tau) = \int_{D_k} T_k(y_1, y_2, \tau) K_k(y_1, \gamma_1) \rho_k(y_1)\, dy_1, \quad k = 1, 2; \tag{18}$$

$$D_1 = \{|y_1| < 1\};\ D_2 = D_{2,1} + D_{2,2};\ D_{2,1} = \{|y_1| < a,\ |y_1| > 1\} \text{ for } 0 < y_2 < a\nu_3(x_1);$$

$$D_{2,2} = \{|y_1| < 1\} \quad \text{for} \quad \xi_3(x_1) < y_2 < a\nu_3(x_1);$$

$$V_k(\gamma_1, \gamma_3, \tau) = \int_{\sigma_k} \Theta_k(\gamma_1, y_3, \tau) K_k(y_3, \gamma_3) \rho_k(y_3)\, dy_3, \quad k = 1, 2; \tag{19}$$

$$\sigma_1 = \{0 < y_3 < 1\},\ \sigma_2 = \sigma_{2,1} + \sigma_{2,2};\ \sigma_{2,1} = \{0 < y_3 < b \quad \text{for} \quad D_{2,1}\};$$

$$\sigma_{2,2} = \{1 < y_3 < b \quad \text{for} \quad D_{2,2}\};$$

$$\Theta_k(\gamma_1, y_3, \tau) = U_k(\gamma_1, y_2, \tau) - e_k \begin{cases} y_3 E_{1,\gamma_1} & \text{at} \quad k = 1; \\ \dfrac{1}{b} y_3 E^{(1)}_{2,\gamma_1} & \text{at} \quad k = 2 \quad \text{for} \quad \sigma_{2,1}; \\ \Phi_{2,\gamma_1} [(1-a) y_3 + (a-b)] (b-1)^{-1} + D_{2,\gamma_1} & \text{at} \quad k = 2 \quad \text{for} \quad \sigma_{2,2}; \end{cases} \tag{20}$$

$$E_{1,\gamma_i} = D_{1,\gamma_i} - \Phi_{1,\gamma_i};\quad E^{(1)}_{2,\gamma_i} = D^{(1)}_{2,\gamma_i} - a\Phi^{(1)}_{2,\gamma_i};$$

$$\Phi_{k,\gamma_i} = [2\alpha_{k,i,\gamma_i} C_{k,\gamma_i} (b_{k,i,\gamma_i} - 1)]^{-1} \left\{ \underset{|k|}{F}\left(b_{k,i,\gamma_i} - 1,\ \frac{1}{2},\ \frac{1}{2}\alpha_{k,i,\gamma_i}\right) - 1 \right\};$$

$$D_{2,\gamma_i} = [C^{(1)}_{2,\gamma_i}]^{-1} \left\{ \frac{2}{3} \underset{|2|}{F}\left(b_{2,i,\gamma_i} - \frac{1}{2},\ \frac{1}{2},\ a_{2,i,\gamma_i}\right) \times \right.$$

$$\times \left[a^{\frac{3}{2}} \underset{|2|}{F}\left(b_{2,i,\gamma_i},\ \frac{5}{2},\ a_{2,i,\gamma_i}\right) - \underset{|2|}{F}\left(b_{2,i,\gamma_i},\ \frac{5}{2},\ \frac{1}{2}\alpha_{2,i,\gamma_i}\right) \right] -$$

$$- 2a [(2b_{2,i,\gamma_i} - 3)\alpha_{2,i,\gamma_i}]^{-1} \underset{|2|}{F}\left(b_{2,i,\gamma_i},\ \frac{3}{2},\ a_{2,i,\gamma_i}\right) \times$$

$$\left. \times \left[\underset{|2|}{F}\left(b_{2,i,\gamma_i} - \frac{3}{2},\ -\frac{1}{2},\ a_{2,i,\gamma_i}\right) - \underset{|2|}{F}\left(b_{2,i,\gamma_i} - \frac{3}{2},\ -\frac{1}{2},\ \frac{1}{2}\alpha_{2,i,\gamma_i}\right) \right] \right\};$$

$$\Phi^{(1)}_{2,\gamma_i} = -[C^{(1)}_{2,\gamma_i}]^{-1} \left\{ [2(b_{2,i,\gamma_i} - 1)\alpha_{2,i,\gamma_i}]^{-1} \times \right.$$

$$\times \underset{|2|}{F}\left(b_{2,i,\gamma_i} - \frac{1}{2},\ \frac{1}{2},\ a_{2,i,\gamma_i}\right) \left[\underset{|2|}{F}\left(b_{2,i,\gamma_i} - 1,\ \frac{1}{2},\ a_{2,i,\gamma_i}\right) - \underset{|2|}{F}\left(b_{2,i,\gamma_i} - 1,\ \frac{1}{2},\ \frac{1}{2},\ \alpha_{2,i,\gamma_i}\right) \right] +$$

$$+ a \underset{|2|}{F}\left(b_{2,i,\gamma_i},\ \frac{3}{2},\ a_{2,i,\gamma_i}\right) \left[a \underset{|2|}{F}\left(b_{2,i,\gamma_i} - \frac{1}{2},\ \frac{3}{2},\ a_{2,i,\gamma_i}\right) - \right.$$

$$\left. \left. - \underset{|2|}{F}\left(b_{2,i,\gamma_i} - \frac{1}{2},\ \frac{3}{2},\ \frac{1}{2}\alpha_{2,i,\gamma_i}\right) \right] \right\};$$

$$F_k(x_1, \tau) = l_\tau - q_{k,\gamma_1};\ l_\tau = \frac{\partial \ln \xi(x_1, \tau)}{\partial \tau};\ a_{k,i} = \frac{1}{2} a^2 \alpha_{k,i};$$

$$q_{k,\gamma_i} = \mu^2_{k,i,\gamma_i} \xi_i^{-2}(\tau);\quad F_{k,\gamma_i} = \frac{d \ln \xi_1(\tau)}{d\tau} D_{k,\gamma_i};$$

$$F^{(1)}_{2,\gamma_i} = D^{(1)}_{2,\gamma_i} \frac{d \ln \xi_1(\tau)}{d\tau};$$

$$C_{k,\gamma_1} = \int^1 y_1^2 \underset{|k|}{F^2}\left(b_{k,1,\gamma_1},\ \frac{3}{2},\ z_{k,\gamma_1}\right) \exp(-z_{k,\gamma_1})\, dy_1;$$

$$C^{(1)}_{2,\gamma_1} = \int_1^a \left[\Delta \underset{|2|}{F}\left(b_{2,1,\gamma_1}, \frac{3}{2}, z_{2,\gamma_1}\right)\right]^2 \exp(-z_{2,\gamma_1})\, dy_1;$$

$$z_{k,\gamma_1} = \frac{1}{2}\alpha_{k,1,\gamma_1} y_1^2, \quad \tau > 0, \quad k = 1,\ 2;$$

for i = 3 instead of a we set b.

We hence assume that the properties of the transformations (18)-(19) postulated below hold uniformly in y_2, τ and γ_1, τ, respectively.

The kernels of the transformation (18) are solutions of the equations

$$a_k \frac{\partial^2 K_k(y_1, \gamma_1)\rho_k(y_1)}{\partial y_1^2} - \xi_1(\tau)\xi_1'(\tau)\left[K_k(y_1, \gamma_1)\rho_k(y_1) - y_1 \frac{\partial K_k(y_1, \gamma_1)\rho_k(y_1)}{\partial y_1}\right] + \mu^2_{k,1} K_k(y_1, \gamma_1)\rho_k(y_1) = 0, \quad k = 1,\ 2, \tag{21}$$

under homogeneous boundary conditions.

Under the conditions

$$\xi_1(\tau)\xi_1'(\tau) = \frac{a_k \rho_k'(y_1)}{y_1\rho_k(y_1)}, \quad k = 1,\ 2, \tag{22}$$

we reduce (21) by the substitutions

$$K_k(y_1, \gamma_1) = W_k(z_{k,1}, \gamma_1)\left(\frac{2z_{k,1}}{\alpha_{k,1}}\right)^{\frac{1}{2}} \exp(-z_{k,1});\ z_{k,1} = \frac{\alpha_{k,1}}{2} y_1^2 \tag{23}$$

to the degenerate hypergeometric equations

$$z_{k,1}\frac{\partial^2 W_k(z_{k,1}, \gamma_1)}{\partial z^2_{k,1}} + \left(\frac{3}{2} - z_{k,1}\right)\frac{\partial W_k(z_{k,1}; \gamma_1)}{\partial z_{k,1}} - b_{k,1}W_k(z_{k,1}, \gamma_1) = 0, \tag{24}$$

where

$$b_{k,1} = 1 - \frac{\mu^2_{k,1}}{2\Lambda_1}; \quad \alpha_{k,1} = \frac{\Lambda_1}{a_k}; \quad k = 1,\ 2; \quad \Lambda_1 = \xi_1(\tau)\xi_1'(\tau), \quad \tau > 0. \tag{25}$$

We determine the weight functions

$$\rho_{k,1}(y_1) = \exp z_{k,1}, \quad k = 1,\ 2; \tag{26}$$

$$\xi_1(\tau) = \beta_1\sqrt{\tau}, \quad \beta_1 = \sqrt{2\Lambda_1} \tag{27}$$

from (22) and (25).

Solving (24) and using (23), we find the normalized kernels of the transformations (18):

$$K_k(y_1, \gamma_1) = y_1[C_{k,\gamma_1}]^{-1}\underset{|k|}{F}\left(b_{k,1,\gamma_1}, \frac{3}{2}, z_{k,\gamma_1}\right)\exp(-z_{k,\gamma_1}), \quad k = 1,\ 2, \tag{28}$$

where k = 2 for $D_{2,2}$ while for $D_{2,1}$

$$K_2(y_1, \gamma_1) = [C^{(1)}_{2,\gamma_1}]^{-1}\Delta\underset{|2|}{F}\left(b_{2,1,\gamma_1}, \frac{3}{2}, z_{2,\gamma_1}\right)\exp(-z_{2,\gamma_1}); \tag{29}$$

here $\underset{|k|}{F}(B, c, z)$ are the degenerate hypergeometric functions

$$C_{k,\gamma_1} = L_{k,\gamma_1}\sum_{n=0}^{\infty}\frac{S_{k,n}\left(b_{k,1,\gamma_1}, \frac{3}{2}, \frac{1}{2}\alpha_{k,1,\gamma_1}\right)}{b_{k,1,\gamma_1} + n}; \tag{30}$$

$$C^{(1)}_{2,\gamma_1} = L^{(2)}_{2,\gamma_1}(1)\sum_{n=0}^{\infty}\left[\frac{\Delta S^{(1)}_{2,n}\left(b_{2,1,\gamma_1}, \frac{3}{2}, \frac{1}{2}\alpha_{2,1,\gamma_1}\right)}{b_{2,1,\gamma_1}+n} - \frac{\Delta S^{(2)}_{2,n}\left(b_{2,1,\gamma_1}, \frac{3}{2}, \frac{1}{2}\alpha_{2,1,\gamma_1}\right)}{b_{2,1,\gamma_1}+n-1}\right]; \qquad (31)$$

$$L_{k,\gamma_1} = \frac{2}{3}\underset{|k|}{F}\left(b_{k,1,\gamma_1}+1, \frac{5}{2}, \frac{1}{2}\alpha_{k,1,\gamma_1}\right)\exp\left(-\frac{1}{2}\alpha_{k,1,\gamma_1}\right);$$

$$L^{(2)}_{k,\gamma_1}(q) = \exp\left(-\frac{1}{2}\alpha_{2,1,\gamma_1}\right)\Big\{a(2b_{2,1,\gamma_1}-1)\times$$

$$\times\underset{|2|}{F}\left(b_{2,1,\gamma_1}+\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\alpha_{2,1,\gamma_1}\right)\underset{|2|}{F}\left(b_{2,1,\gamma_1}, \frac{3}{2}, a_{2,1,\gamma_1}\right)-$$

$$-\left[\frac{2}{3}q_{2,\gamma_1}b_{2,1,\gamma_1}\underset{|2|}{F}\left(b_{2,1,\gamma_1}+1, \frac{5}{2}, \frac{1}{2}\alpha_{2,1,\gamma_1}\right)+\right.$$

$$\left.+\underset{|2|}{F}\left(b_{2,1,\gamma_1}, \frac{3}{2}, \frac{1}{2}\alpha_{2,1,\gamma_1}\right)\right]\underset{|2|}{F}\left(b_{2,1,\gamma_1}-\frac{1}{2}, \frac{1}{2}, a_{2,1,\gamma_1}\right)\Big\};$$

$$a_{k,1} = \frac{1}{2}\alpha_{k,1}a^2;$$

$$\Delta\underset{|2|}{F}\left(b_{2,1}, \frac{3}{2}, z_{2,1}\right) = \begin{vmatrix} y_1\underset{|2|}{F}\left(b_{2,1}, \frac{3}{2}, z_{2,1}\right) & \underset{|2|}{F}\left(b_{2,1}-\frac{1}{2}, \frac{1}{2}, z_{2,1}\right) \\ a_1\underset{|2|}{F}\left(b_{2,1}, \frac{3}{2}, a_{2,1}\right) & \underset{|2|}{F}\left(b_{2,1}-\frac{1}{2}, \frac{1}{2}, a_{2,1}\right)\end{vmatrix}. \qquad (32)$$

Hence, by replacing the determinant $\underset{|2|}{F}$ in the j-th column by $S^{(j)}_{2,n}$ with the same arguments, we obtain $\Delta S^{(j)}_{2,n}(b_{2,1}, 3/2, z_{2,1})$. Here $S_{2,n}(b, c, x)$ is a partial sum of the degenerate hypergeometric series for the function $\underset{|2|}{F}(b, c, x)$.

The nontrivial solutions of the Sturm–Liouville problems under consideration are evidently possible only for values of $\mu_{k,1} = \mu_{k,1}(\alpha_{k,1})$ satisfying the equations

$$\underset{|k|}{F}\left(b_{k,1}, \frac{3}{2}, \frac{1}{2}\alpha_{k,1}\right) = 0, \quad k=1, 2, \qquad (33)$$

where k = 2 for $D_{2,2}$ while for $D_{2,1}$

$$\Delta\underset{|2|}{F}\left(b_{2,1}, \frac{3}{2}, \frac{1}{2}\alpha_{2,1}\right) = 0. \qquad (34)$$

The eigenvalues of these problems form complex spectra which depend on the position and rate of displacement of the boundary $S_{\tau,1}$, as well as on the inertial properties of the ground.

By using the transformations (18)-(19) we reduce the problem (11)-(17) to the form

$$\frac{dV_k(\gamma_1, \gamma_3, \tau)}{d\tau} + (q_{k,\gamma_1}+q_{k,\gamma_3})V_k(\gamma_1, \gamma_3, \tau) = P_k(\gamma_1, \gamma_3, \tau), \quad k=1, 2; \qquad (35)$$

$$V_1(\gamma_1, \gamma_3, 0) = B^{(1,1)}_{\gamma_1,\gamma_3}; \qquad (36)$$

$$V_2(\gamma_1, \gamma_3, 0) = B^{(2,i)}_{\gamma_1,\gamma_3} \quad \text{for} \quad \sigma_{2,i}, \quad i=1, 3; \qquad (37)$$

$$P_1(\gamma_1, \gamma_3, \tau) = t_e\left[\frac{1}{2}D_{1,\gamma_3}E_{1,\gamma_1}F_1(x_1, \tau) - \Phi_{1,\gamma_3}F_{1,\gamma_1}\right];$$

$$P_2(\gamma_1, \gamma_3, \tau) = \begin{cases} \frac{1}{2b}e_2D_{2,\gamma_3}E^{(1)}_{2,\gamma_1}F_2(x, \tau) - \Phi_{2,\gamma_3}F^{(1)}_{2,\gamma_1} & \text{for } \sigma_{2,1}; \\ \frac{t_0}{2(b-1)}\Phi_{2,\gamma_1}D^{(1)}_{2,\gamma_3}F_2(x_1, \tau) - \left[\left(\frac{a-b}{b-1}\Phi_{1,\gamma_1} + D_{2,\gamma_1}\right)q_{2,\gamma_1} + F_{2,\gamma_1}\right]e_2\Phi_{2,\gamma_3} & \text{for } \sigma_{2,2}; \end{cases}$$

$$B^{(1,1)}_{\gamma_1,\gamma_3} = t_e\left[\frac{1}{2}D_{1,\gamma_3}E_{1,\gamma_1} + D_{1,\gamma_1}\Phi_{1,\gamma_3}\right] + \Phi_{1,\gamma_1}\Phi_{1,\gamma_3}\Delta t_e;$$

$$B^{(2,i)}_{\gamma_1,\gamma_3} = \begin{cases} \left(\frac{1}{2b}D_{2,\gamma_3} - \Phi_{2,\gamma_3}\right)e_2E^{(1)}_{2,\gamma_1} & \text{for } i=1; \\ \frac{t_0}{1-b}\Phi_{2,\gamma_1}\left(\frac{1}{2}D^{(1)}_{2,\gamma_3} - b\Phi^{(1)}_{2,\gamma_3}\right) & \text{for } i=2; \end{cases}$$

$$b_3 = 1 - \frac{\mu^2_{k,3}}{2\Lambda_3}; \quad \alpha_{k,3} = \frac{\Lambda_3}{a_n}; \quad \Lambda_3 = \frac{\partial\xi^2(x,\tau)}{\partial\tau}; \tag{38}$$

$\mu_{k,3}$ satisfies equations of the form (33) for k = 1 and for $\sigma_{2,1}$ for k = 2 (with $\alpha_{2,1}$ replaced by $\alpha_{2,3}b^2$) and (34) for $\sigma_{2,2}$ (with a replaced by b). The kernels of the transformations (19) are determined by (28) for k = 1 and for $\sigma_{2,1}$ for k = 2 (with $\alpha_{2,1}$ replaced by $\alpha_{2,3}b^2$ in C_{2,γ_1}) and for $\sigma_{2,2}$ by means of (29) with a replaced by b; the weight functions are determined by means of (26) with γ_1 replaced by γ_3, where

$$\xi_2(\tau) = \beta_2\sqrt{\tau}, \tag{39}$$

and $\beta_2 = \sqrt{2\Lambda_2}$; $\Lambda_2 = \xi_2(\tau)\xi_2'(\tau)$, $\tau > 0$.

Therefore, the passage from the plane (x_1, x_2) to the plane (y_1, y_3) generates a parameter Λ_i connecting the position of the moving boundary to the time during the solution. The introduction of these parameters affords a possibility of sampling and normalizing the minimal system of eigenfunctions in the continuous spectrum in the domains under consideration and, thereby, contributes to the construction of a general solution of the problem.

Having solved the problem (35)-(37) and having realized the inverse transformations, we obtain the solution of the problem (11)-(17) in the form of the rapidly converging series

$$T_1(y_1, y_2, \tau) = \sum_{\gamma_1=1}^{\infty}\sum_{\gamma_3=1}^{\infty} A^{(1,1)}_{\gamma_1,\gamma_3}\prod_{i=1,3}\frac{x_i}{\tau}\underset{|1|}{F}\left(b_{1,i,\gamma_i}, \frac{3}{2}, \frac{x_i^2}{4a_1\tau}\right)\exp\left(-\frac{x_i^2}{4a\tau}\right); \tag{40}$$

$$T_2(y_1, y_2, \tau) = \sum_{\gamma_1=1}^{\infty}\sum_{\gamma_3=1}^{\infty}\sum_{i=1,3} A^{(2,i)}_{\gamma_1,\gamma_3}\underset{|2|}{F}\left(b_{2,i,\gamma_i}, \frac{3}{2}, \frac{x_i^2}{4a_2\tau}\right)\times$$

$$\times\Delta\underset{|2|}{F}\left(b_{2,i,\gamma_i}, \frac{3}{2}, \frac{x_i^2}{4a_2\tau}\right)\exp\left(-\frac{x_i^2}{4a_2\tau}\right), \tag{41}$$

where

$$A^{(k,i)}_{\gamma_1,\gamma_3} = (2\Lambda_{i,\gamma_i})^{-\frac{1}{2}}(B^{(k,i)}_{\gamma_1,\gamma_3}E^{(k,1)}_{\gamma_1,\gamma_3} + C^{(k,i)}_{\gamma_1,\gamma_3});$$

$$C^{(1,1)}_{\gamma_1,\gamma_3} = \frac{t_e}{\Delta^{(1)}_{\gamma_1,\gamma_3}}\left[\frac{1-\delta_{1,\gamma_1}}{2}D_{1,\gamma_3}E_{1,\gamma_3} - D_{1,\gamma_1}\Phi_{1,\gamma_3}\right];$$

$$C^{(2,1)}_{\gamma_1,\gamma_3} = \frac{e_2}{\Delta^{(2)}_{\gamma_1,\gamma_3}}\left[\frac{1-\delta_{2,\gamma_1}}{2b}D_{2,\gamma_3}E^{(1)}_{2,\gamma_3} - D^{(1)}_{2,\gamma_1}\Phi_{2,\gamma_3}\right];$$

$$C^{(2,2)}_{\gamma_1,\gamma_3} = \frac{e_2}{\Delta^{(2)}_{\gamma_1,\gamma_3}}\left\{\frac{1-\delta_{2,\gamma_1}}{2(b-1)}(1-a)D^{(1)}_{2,\gamma_3}\Phi_{2,\gamma_1} + \right.$$

$$\left. + \left[\left(D_{2,\gamma_1} + \frac{a-b}{b-1}\Phi_{2,\gamma_1}\right)\delta_{2,\gamma_1} + D_{2,\gamma_1}\right]\Phi_{2,\gamma_3}\right\};$$

$$E^{(k,1)}_{\gamma_1,\gamma_3} = \left(\frac{l_0}{\xi_1(\tau)}\right)^{\delta_{k,\gamma_1}}\left(\frac{\xi_0}{\xi_2(\tau)}\right)^{\delta_{k,\gamma_3}};$$

$$\delta_{k,i} = \frac{\mu^2_{k,i,\gamma_i}}{\Lambda_i}; \quad \Delta^{(k)}_{\gamma_1,\gamma_3} = \sum_{i=1,3}\delta_{k,\gamma_i}; \quad l = \begin{cases}1 & \text{for } i=3, \\ 3 & \text{for } i=1;\end{cases}$$

$$\tau > 0, \quad k = 1, 2.$$

Using these solutions, we obtain the equation

$$A\frac{\partial\xi(x_1,\tau)}{\partial\tau}+B_1\frac{\partial\xi(x_1,\tau)}{\partial x_1}=B_3 \tag{42}$$

from condition (6) on $S_{\tau,1}$, where

$$B_3=\frac{1}{\xi(x_1,\tau)}\sum_{\gamma_1=1}^{\infty}\sum_{\gamma_2=1}^{\infty}\left\{\frac{\lambda_1x_1}{\sqrt{\tau}}A^{(1,1)}_{\gamma_1,\gamma_2}B^{(1)}_{1,\gamma_1}\underset{|1|}{F}\left(b_{1,1,\gamma_1},\ \frac{3}{2},\ \frac{x_1^2}{4a_1\tau}\right)\times\right.$$

$$\times\exp\left(-\frac{x_1^2}{4a_1\tau}\right)-\lambda_2e_2\left[A^{(2,1)}_{\gamma_1,\gamma_2}B^{(1)}_{2,\gamma_2}\,\Delta\underset{|2|}{F}\left(b_{2,1,\gamma_1},\ \frac{3}{2},\ \frac{x_1^2}{4a_2\tau}\right)+\right.$$

$$\left.\left.+\frac{x_1}{\sqrt{\tau}}A^{(2,2)}_{\gamma_1,\gamma_2}B^{(2)}_{2,\gamma_2}\underset{|2|}{F}\left(b_{2,1,\gamma_1},\ \frac{3}{2},\ \frac{x_1^2}{4a_2\tau}\right)\right]\exp\left(-\frac{x_1^2}{4a_2\tau}\right)\right\};$$

$$B_1=\frac{B_0}{\xi_1(\tau)};\ B_0=2\eta w+\lambda_2ae_2\sum_{\gamma_1=1}^{\infty}\sum_{\gamma_2=1}^{\infty}A^{(2,1)}_{\gamma_1,\gamma_2}\underset{|2|}{F}\left(b_{2,1,\gamma_2},\ \frac{3}{2},\ a_{2,3,\gamma_2}\right);$$

$$B^{(2)}_{2,\gamma_i}=L^{(1)}_{2,\gamma_i}(\alpha_{2,i,\gamma_i});$$

$$B^{(1)}_{k,\gamma_i}=\alpha_{k,i,\gamma_i}L_{k,\gamma_i}\begin{cases}1 & \text{for}\quad k=1;\\ \sqrt{\Lambda_{i,\gamma_i}} & \text{for}\quad k=2;\end{cases}$$

$$\eta=\begin{cases}1 & \text{for}\quad x_1>0;\\ -1 & \text{for}\quad x_1<0;\\ 0 & \text{for}\quad x_1=0;\end{cases}$$

$$w=\lambda_1e_1-\lambda_2e_2.$$

Therefore, the family of lines $S_{\tau,1}$ of the parameter τ, which characterizes the position of the front of the freezing ground, is determined by a nonstationary field of directions formed by the quantities A, B_1, and B_3 on the plane (x_1, x_3).

Having solved (42) under the conditions (8), we find

$$x_2=\frac{1}{A}\sum_{\gamma_1=1}^{\infty}\sum_{\gamma_2=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{\lambda_1}{\sqrt{a_1}}\Psi^{(1)}_{1,n}B^{(1)}_{1,\gamma_1}P^{(1,1)}_{\gamma_1,\gamma_2}\left(\frac{x_1^2}{4a_1\tau}\right)-\right.$$

$$-\frac{\lambda_2}{\sqrt{a_2}}e_2\left[\left(\Psi^{(1)}_{2,n}P^{(2,1)}_{\gamma_1,\gamma_2}\left(\frac{x_1^2}{4a_2\tau}\right)\underset{|2|}{F}\left(b_{2,3,\gamma_2}-\frac{1}{2},\ \frac{1}{2},\ a_{2,3,\gamma_2}\right)-\right.\right.$$

$$\left.\left.\left.-a\varphi^{(1)}_{2,n}Q^{(2,1)}_{\gamma_1,\gamma_2}\left(\frac{x_1^2}{4a_2\tau}\right)\underset{|2|}{F}\left(b_{2,3,\gamma_2},\ \frac{3}{2},\ a_{2,3,\gamma_2}\right)\right]B^{(1)}_{2,\gamma_2}+\Psi^{(1)}_{2,n}B^{(2)}_{2,\gamma_2}P^{(2,2)}_{\gamma_1,\gamma_2}\left(\frac{x_1^2}{4a_2\tau}\right)\right]\right\}, \tag{43}$$

where

$$P^{(k,i)}_{\gamma_1,\gamma_2}\left(\frac{x_1^2}{4a_k\tau}\right)=\frac{[Ax_1-B^{(1)}_0\xi_1(\tau)]^2}{(A-B^{(1)}_0)^2}a^{(k,i)}_{\gamma_1,\gamma_2}\left(n,\ \frac{1}{2}\alpha_{k,1,\gamma_1}\right)-x_1^2a^{(k,i)}_{\gamma_1,\gamma_2}\left(n,\ \frac{x_1^2}{4a_k\tau}\right);$$

$$Q^{(k,i)}_{\gamma_1,\gamma_2}\left(\frac{x_1^2}{4a_k\tau}\right)=\frac{Ax_1-B^{(1)}_0\xi_1(\tau)}{A-B^{(1)}_0}a^{(k,i)}_{\gamma_1,\gamma_2}\left(n,\ \frac{1}{2}\alpha_{k,1,\gamma_1}\right)-x_1a^{(k,i)}_{\gamma_1,\gamma_2}\left(n,\ \frac{x_1^2}{4a_k\tau}\right);$$

$$B^{(1)}_0=C^{(1)}_0-\sum_{\gamma_1=1}^{\infty}2\eta\frac{w}{\Lambda_{1,\gamma_1}};\qquad \Psi^{(i)}_{k,n}=\frac{(b_{k,i,\gamma_i})_n}{\left(\frac{3}{2}\right)_n n!};$$

$$\varphi^{(i)}_{k,n}=\frac{\left(b_{k,i,\gamma_i}-\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n n!};$$

$$C^{(i)}_0=\lambda_2e_2a\sum_{\gamma_1=1}^{\infty}\sum_{\gamma_2=1}^{\infty}\Lambda^{-1}_{1,\gamma_i}A^{(2,1)}_{\gamma_1,\gamma_2}B^{(i)}_{2,\gamma_i}\underset{|2|}{F}\left(b_{2,l,\gamma_l},\ \frac{3}{2},\ a_{2,l,\gamma_l}\right);$$

$$a^{(k,i)}_{\gamma_1,\gamma_2}(n,\ q)=B^{(k,i)}_{\gamma_1,\gamma_2}E^{(k,i)}_{\gamma_1,\gamma_2}\gamma\left(n+\frac{1}{2}[\Delta^{(k)}_{\gamma_1,\gamma_2}-1],\ q\right)+C^{(k,i)}_{\gamma_1,\gamma_2}\gamma\left(n-\frac{1}{2},\ q\right);$$

$\gamma(n, q)$ is the incomplete gamma function, and $\xi_1(\tau)$ is determined from (25).

The expression (43) describes the shape and the motion law for the $S_{\tau,1}$ boundary for $|x_1| < \xi_1(\tau)$; however, it contains the parameters Λ_i. They connect the magnitude of the displacement of the boundary $S_{\tau,1}$ in the directions of the axes Ox_1 and Ox_2 to the time τ. Taking this into account and using the solutions (40)-(41), we obtain from condition (6)

$$A + B_0^{(1)} = 0, \quad A + C_0^{(3)} = 0. \tag{44}$$

We determine the values of Λ_i and $\mu_{k,i}$ from the system (43)-(44) and the characteristic equations. Values of these parameters which satisfy (43)-(44) can be set up in engineering practice by the method of sampling, by using tables of zeroes of the characteristic functions as a function of $\alpha_{k,i}$, as well as by appropriate graphs [7-9]. Using these results, we finally set up the shape and motion law for the boundary $S_{\tau,1}$ from (47), and the nature of the temperature distribution in zones I and II from (40)-(41) with (10) taken into account; and the solution of the problem is thereby completed.

NOTATION

t_0 and t_e, temperature of the ground and the heat carrier; λ_k and a_k, heat conductivity and thermal diffusivity coefficients; σ, latent heat of crystallization of water; w_w, humidity of the ground; γ_w, water density.

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